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Quadratic Spaces with Trivial Arf Invariant

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In [7], a classification of quadratic spaces of rank 3 and 4 with trivial discriminant over a domain R in which 2 is invertible was given in terms of projective modules over central separable R-algebras of rank 4. We extend here this classification to quadratic spaces of rank 4 over any commutative ring. For this we replace the condition on the discriminant by the condition that the Arf invariant of the space is trivial. If $\frac{1}{2} \in R$ and R is a domain, we show that both conditions are equivalent. A main step is to verify that a quadratic space of even rank has trivial Arf invariant if and only if its Clifford algebra has an idempotent which "determines the gradation." Using this idempotent, we associate to any quadratic space (V, q) of rank $4n$ a central separable R-algebra A with involution and a projective A-module of rank one P, which carries a hermitian structure over A. The pairs (A, P) classify the quadratic spaces (V, q). If (V, q) has rank 4 over R, then A is central separable of rank 4 and the hermitian structure on P induces a quadratic structure on P with values in R. This quadratic structure is in fact the reduced norm of P as defined in [7].

If $\mathrm{Pic}(R)$ has no 2-torsion and A is any central separable R-algebra of rank 4, then any projective A-module of rank one is a hermitian space over A, hence carries a quadratic form. If the central separable algebra A is the extension of a quadratic algebra S (like a quaternion algebra), then the module P is also a hermitian space of rank two over S. Two such modules are isomorphic if and only if the corresponding hermitian spaces over S are

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isomorphic up to a unit of R. This generalizes results of Parimala [12] and Kopeiko [8].

This paper is nearly self-contained. In particular we have recalled many results on quadratic forms. We have included proofs of some results which we did not find with the corresponding generality in the literature. As usual, R denotes a commutative ring with 1 and unadorned tensor products are taken over R. For simplicity we shall always assume that projective modules have constant rank. We thank M. Ojanguren for many helpful comments on a first version of this paper.

1. Quadratic Spaces

We recall some basic definitions on quadratic spaces over commutative rings. Details and proofs can be found in [1] or [9]. Let R be a commutative ring and V be a finitely generated projective R-module. A *quadratic form* on V is a map $q: V \to R$ such that

(1) $q(\lambda x) = \lambda^2 q(x)$ for all $x \in V$, $\lambda \in R$;

(2) $b_q(x, y) = q(x+y) - q(x) - q(y)$ defines on V a bilinear form.

The pair (V, q) is called a *quadratic space* over R if the form b_q is *nonsingular*, that is, it induces an isomorphism

$$\varphi_q: V \simeq \mathrm{Hom}_R(V, R).$$

We have an obvious notion of an isomorphism between quadratic modules. Such an isomorphism will be called an *isometry*.

Example (1.1). Let V be a finitely generated projective R-module and $V^* = \mathrm{Hom}_R(V, R)$. We define on $V \oplus V^*$ a quadratic form by $q_V(x + x^*) = x^*(x)$ for all $x \in V$ and $x^* \in V^*$. The associated bilinear form is nonsingular hence $(V \oplus V^*, q_V)$ is a quadratic space and is called the *hyperbolic space* associated with V. It will be denoted by $H(V)$.

The *Clifford algebra* $C = C(V, q)$ of a quadratic space (V, q) is the quotient of the tensor algebra TV of the R-module V by the two sided ideal generated by all elements of the form $x \otimes x - q(x)$, $x \in V$. The canonical map $V \to T^1 V$ induces an injective homomorphism of R-modules $i: V \to C$. By definition of C, we have $i(x)^2 = q(x)$ for all $x \in V$ and the pair (C, i) is universal with respect to this property: for any R-algebra E and any R-module homomorphism $\rho: V \to E$ such that $\rho(x)^2 = q(x)$, $x \in V$, in E, there is a unique R-algebra homomorphism $\rho': C(V, q) \to E$ such that $\rho' \circ i = \rho$. In particular any isometry $\tau: (V, q) \to (V', q')$ induces an isomorphism $\tau': C(V, Q) \simeq C(V', q')$. The algebra $C(V, q)$ has a $\mathbb{Z}/2\mathbb{Z}$-gradation

$C = C_0 \oplus C_1$ induced by the obvious $\mathbb{Z}/2\mathbb{Z}$-gradation of TV and the isomorphism τ' is graded. We have the following structure theorem for Clifford algebras of quadratic spaces [9]:

THEOREM (1.2). *Let $C = C_0 \oplus C_1$ be the Clifford algebra of the quadratic space (V, q). Then*:

(1) *If* $\operatorname{rank}_R V$ *is even, then C is a central separable R-algebra. The center $Z(C_0)$ of C_0 is a separable R-algebra and is projective of rank two as an R-module. Furthermore C_0 is central separable over $Z(C_0)$.*

(2) *If* $\operatorname{rank}_R V$ *is odd, then C_0 is a central separable R-algebra. The center $Z(C)$ of C is a separable R-algebra and is projective of rank two as an R-module. Furthermore C is central separable over $Z(C)$.*

We recall the definition of a *separable algebra*. For any R-algebra A, let A^0 be the opposite algebra. Clearly any R-algebra A is a left $A \otimes A^0$-module and A is called *separable* if A is $A \otimes A^0$-projective. For any R-algebra A, the center $Z(A)$ of A is defined as $Z(A) = \{x \in A/ax = xa, \ \forall a \in A\}$. The unit map $R \to A$ has image in $Z(A)$ and A is called *central* if $Z(A) = R$. An algebra which is central and separable is called central separable or an *Azumaya algebra*. We refer, for example, to [6] for a study of these algebras. We shall see more examples later.

EXAMPLE (1.3). Let $H(V)$ be a hyperbolic space. Then $C(H(V)) \simeq \operatorname{End}_R(\Lambda V)$ as graded algebras, where ΛV is the exterior algebra of V. We take on ΛV the gradation given by the degree modulo two and on $\operatorname{End}_R(\Lambda V)$ the induced "chess-board" gradation. That is we have $C_0 \simeq \operatorname{End}_R((\Lambda V)^0) \times \operatorname{End}_R((\Lambda V)^1)$ and

$$C_1 \simeq \operatorname{Hom}_R((\Lambda V)^0, (\Lambda V)^1) \times \operatorname{Hom}_R((\Lambda V)^1, (\Lambda V)^0).$$

In particular we see that $Z(C_0) = R \times R$.

2. QUADRATIC ALGEBRAS

We shall call an R-algebra S *quadratic* if S is a separable R-algebra and is a projective R-module of rank two. We recall the following properties of quadratic algebras (see [1] or [9]): R is a direct summand of S as R-module, S is commutative, S possesses a unique nontrivial automorphism $\sigma = \sigma_S$ with $\sigma^2 = \mathrm{Id}$ and $\mathrm{Fix}(\sigma) = R$. We shall call σ the *canonical involution* of S. We use σ to define the *norm* N_S and the *trace* T_S of S:

$$N_S(x) = x\sigma(x), \qquad T_S(x) = x + \sigma(x), \qquad \text{for all } x \in S.$$

Let $X(S) = \ker T_S$, then $S = \mathrm{Re} \oplus X(S)$ as R-module, where $e \in S$ is such that $T_S(e) = 1$. In particular $X(S)$ is a projective R-module of rank one. The norm N_S is a quadratic form on S and the associated bilinear form is nonsingular. Hence the pair (S, N_S) is a quadratic space, called the *norm form* of S.

EXAMPLE (2.1). (See [1] or [9].) If S is a free R-module, then $S \simeq R[t]/(t^2 - at - b)$, where the polynomial $t^2 - at - b$ is separable, which means that its discriminant $a^2 + 4b$ is a unit of R. If z denotes the class of t in S, then $T_S(z) = a$ and $N_S(z) = -b$, hence $\sigma(z) = a - z$. The bilinear form associated to the norm of S is given by the matrix

$$\begin{pmatrix} 2 & a \\ a & -2b \end{pmatrix}$$

with respect to the basis $\{1, z\}$ of S. Furthermore $X(S) = R(1 - 2z)$. If R is a local ring, then by a change of variable, we can obtain that $a = 1$; hence $S = R[t]/(t^2 - t - b$ and $1 + 4b$ is a unit.

EXAMPLE (2.2). (See [1] or [9].) If $2 = 0$ in R, then $\ker T_S = \mathrm{Fix}(\sigma_S) = R$. Hence any quadratic R-algebra is a free R-module. By Example (2.1), $S = R[t]/(t_2 - at - b)$ and a is a unit. Hence, as in the local case, we can assume that $a = 1$.

The multiplication map $\mu: S \otimes S \to S$ gives by restriction a map $X(S) \otimes X(S) \to S$ which takes actually values in $\mathrm{Fix}(\sigma_S) = R$. By checking it locally (use (2.1)) we see that μ induces an isomorphism $\mu: X(S) \otimes X(S) \simeq R$. A pair (L, α), where L is a projective R-module of rank one and α is an isomorphism $L \otimes L \simeq R$ is called a *discriminant module.* The tensor product induces the structure of an abelian group on the set of isomorphism classes of discriminant modules. This group is called the *discriminant group* of R and is denoted by $\mathrm{Disc}(R)$ (see [3]). The set of isomorphism classes of quadratic R-algebras can also be endowed with the structure of an abelian group: let S_1 and S_2 be quadratic extensions of R with canonical involutions σ_1 and σ_2. We define $[S_1] \cdot [S_2] = [\mathrm{Fix}(\sigma_1 \otimes \sigma_2)]$, where $[S]$ denotes the isomorphism class of S. This group is called the *group of quadratic extensions* of R and is denoted by $\Delta(R)$. The neutral element of R is the class of $R \times R$.

PROPOSITION (2.3). *The map $S \mapsto (X(S), \mu)$ induces a group homomorphism $d: \Delta(R) \to \mathrm{Disc}(R)$. If 2 is invertible in R, d is an isomorphism and if $2 = 0$ in R, d is the zero map.*

Proof. See Proposition (2.4.15) of [9].

Let (V, q) be a quadratic space over R and let $C = C_0 \oplus C_1$ be the Clifford algebra of (V, q). We define

$$D(V, q) = Z(C_0) \quad \text{if} \quad \operatorname{rank}_R V \text{ is even}$$
$$= Z(C) \quad \text{if} \quad \operatorname{rank}_R V \text{ is odd.}$$

By (1.2), $D(V, q)$ is a quadratic algebra. Its class in $\Delta(R)$ is called the *Arf invariant* of (V, q) and we shall denote it by $a(V, q)$.

EXAMPLE (2.4). Let R be a local ring and let (V, q) be a quadratic space over R of even rank. Then (V, q) is the orthogonal sum of n planes: there exists a basis $\{e_1, ..., e_{2n}\}$ of V such that

$$q\left(\sum_{i=1}^{2n} \lambda_i e_i\right) = \sum_{i=1}^{n} (\alpha_i \lambda_i^2 + \lambda_i \lambda_{i+n} + \beta_i \lambda_{i+n}^2)$$

where $1 - 4\alpha_i\beta_i$ and α_i, $1 \leqslant i \leqslant n$, are units of R. The elements e_i and e_{i+n} generate the ith plane, e_i is such that $q(e_i) = \alpha_i$ is a unit of R and $b_q(e_i, e_{i+n}) = 1$. We have $D(V, q) = R \oplus Rz$ with $z^2 = z + d$, where

$$z = \sum_{i=1}^{n} (-2)^{i-1} s_i(e_1 e_{1+n}, ..., e_n e_{2n}),$$

$$d = \sum_{i=1}^{n} 4^{i-1} s_i(-\alpha_1\beta_1, ..., \alpha_n\beta_n)$$

and s_i is the ith symmetric elementary function in n letters (see [1, p. 54] or [10]). Furthermore the element z satisfies $zx + xz = x$ for all $x \in V$ (see the proof of Théorème 3 in [10]).

Let now (V, q) be of odd rank. By [10, p. 276], $2 \notin \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R and by [10, Théorème 1], there exists an orthogonal basis $\{e_1, ..., e_n\}$ for (V, q): we have

$$q\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \alpha_i \lambda_i^2$$

where α_i, $1 \leqslant i \leqslant n$, are units of R and $D(V, q) = R \oplus Rz$ with $z^2 = d$, where

$$z = e_1 \cdot \cdots \cdot e_n \qquad \text{and} \qquad d = (-1)^{m(m-1)/2} \alpha_1 \cdots \alpha_n.$$

For any Clifford algebra $C = C(V, q)$, let α be the *canonical automorphism* of C and let $\sigma = \sigma_C$ be the *canonical involution* of C. We recall that α is the automorphism of C such that $\alpha(x) = -x$ for $x \in V$ and that σ is such that $\sigma(xy) = \sigma(y)\,\sigma(x)$ and $\sigma(x) = x$ for $x \in V$.

LEMMA (2.5). *If the rank of V is odd, then the canonical involution of the quadratic algebra $D = D(V, q)$ is the restriction of α to D. If* $\operatorname{rank}_R V \equiv 2(4)$ *then the canonical involution of D is the restriction of σ_C to D. If* $\operatorname{rank}_R V \equiv O(4)$, *then the restriction of σ_C to D is the identity.*

Proof. If suffices to verify (2.5) locally. Then the first claim is an easy consequence of (2.4). We assume now that V has even rank. By Example (2.4), (V, q) is the orthogonal sum of n planes (V_i, q_i), $1 \leqslant i \leqslant n$, and we prove (2.5) by induction on n. The case $n = 1$ is easily checked directly. For the general case, we assume that $(V, q) \simeq (V', q') \perp (V'', q'')$ with V' and V'' of even rank. Then by [1, p. 26], we have $D(V, q) \simeq (D' \otimes D'')^{\sigma' \otimes \sigma''}$, where $D' = D(V', q')$, $D'' = D(V'', q'')$, σ' is the canonical involution of D' and σ'', the canonical involution of D''. If z' (resp. z'') is the generator of D' (resp. D'') given in (2.4), then $z = 1 \otimes z'' + z' \otimes 1 - 2z' \otimes z''$ is the generator of D. Furthermore we have that $1 \otimes \sigma'' = \sigma' \otimes 1$ on D and this automorphism is the canonical involution of D. Now (2.5) is easy to prove by induction, considering all possible cases for the ranks of V' and V''.

We describe now the Clifford algebra of a norm form (S, N_S), S a quadratic algebra. For any quadratic algebra S and any unit c of R we define an algebra $A = (c, S]$ of rank 4 as $A = S \oplus uS$, $u^2 = c$ and $xu = u\sigma_S(x)$, for $x \in S$. The algebra A is central separable over R ([1, p. 27] or Section 3) and is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with $A_0 = S$ and $A_1 = uS$.

PROPOSITION (2.6). (See [1].) *For any quadratic R-algebra S we have $C(S, N_S) \simeq (1, S]$ as graded algebras. In particular we have $a(S, N_S) = [S]$ in $\Delta(R)$.*

Proof. Let $\rho: S \to A = (1, S]$ be given by $x \mapsto xu$, $x \in S$. It is immediate that $\rho(x))^2 = x\sigma_S(x) = N_S(x)$; hence, by the universal property of the Clifford algebra we have an homomorphism $\rho': C(S, N_S) \to A$ of algebras. Both algebras are central separable and of the same rank. The claim then follows from the following well-known property of central separable algebras:

LEMMA (2.7). *Let C and D be central separable R-algebras of the same rank and let $\varphi: C \to D$ be an homomorphism of algebras. Then φ is an isomorphism.*

COROLLARY (2.8). (See [1].) *Let S_1 and S_2 be quadratic R-algebras. Then the norm forms (S_1, N_{S_1}) and (S_2, N_{S_2}) are isometric if and only if the algebras S_1 and S_2 are isomorphic.*

Proof. If the algebras are isomorphic, then it follows from the uniqueness of the canonical involutions that the norm forms are isometric. The converse follows from (2.6) since any isometry induces a graded isomorphism of the Clifford algebras.

COROLLARY (2.9). *Let S be a quadratic algebra. Then $S \simeq R \times R$ if and only if (S, N_S) is isometric to an hyperbolic plane.*

Proof. On $R \times R$ the involution is given by $\sigma(a, b) = (b, a)$. Hence $U = R \times 0$ is a totally isotropic subspace such that $U^{\perp} = U$. Then by [1, p. 17], (S, N_S) is isometric to the hyperbolic plane $H(U)$. Conversely by (1.3), $a(S, N_S) = 0$ in $\Delta(R)$ if N_S is hyperbolic. Then by (2.6) we have $a(S, N_S) = [S]$, hence $S \simeq R \times R$.

We prove now the following localization property of $\Delta(R)$:

THEOREM (2.10). *For any domain R, the canonical map*

$$\Delta(R) \to \prod_{\mathfrak{m} \in \max(R)} \Delta(R_{\mathfrak{m}})$$

is injective.

Proof. Let $[S]$ be in the kernel. Let k be the quotient field of R, then $[S \otimes k] = 0$ in $\Delta(k)$. By (2.9) then $(S \otimes k, N_{S \otimes k})$ is a hyperbolic plane over k. Let L_1 and L_2 be two isotropic lines of $N_{S \otimes k}$, $S \otimes k \simeq L_1 \oplus L_2$, and let $I_i = S \cap L_i$, $i = 1, 2$. Clearly $I_1 \cap I_2 = 0$; hence, if we can show that $S = I_1 + I_2$, then S is the direct sum of two totally isotropic subspaces and S will be hyperbolic. For each $\mathfrak{m} \in \max(R)$, $S_{\mathfrak{m}}$ is hyperbolic, say $S_{\mathfrak{m}} = J_1 \oplus J_2$, J_i isotropic. Since $(I_i)_{\mathfrak{m}}$ is maximal isotropic, we must have $(I_1)_{\mathfrak{m}} = J_1$ (or J_2), hence locally $S_{\mathfrak{m}} = (I_1)_{\mathfrak{m}} + (I_2)_{\mathfrak{m}}$. This implies that $S = I_1 + I_2$.

Remark (2.11). The corresponding statement for Disc: "if R is a domain, then $\mathrm{Disc}(R) \to \prod_{\mathfrak{m} \in \max(R)} \mathrm{Disc}(R_{\mathfrak{m}})$ is injective" was proved by Bass in [3]. Bass also gave there an example showing that if R is not a domain, the result need not be correct. Since in this example $\frac{1}{2} \in R$, it follows from (2.3) that the same example also works for $\Delta(R)$. Hence (2.10) is not necessarily injective if R is not a domain.

Another invariant of a quadratic space (V, q) is the *signed discriminant.* Let $D_1(V, q)$ be the submodule of $C = C(V, q)$ defined by $D_1(V, q) = \{y \in C / \alpha(y)\, x = -xy\}$, where α is the canonical automorphism C (see Section 2). By [1, p. 42], $D_1(V, q)$ is always contained in $D(V, q)$ and the multiplication map μ of C induces an isomorphism $D_1(V, q) \otimes D_1(V, q) \simeq R$. Hence $D_1(V, q)$ is a discriminant module and its class in $\mathrm{Disc}(R)$ is called the *signed discriminant* of (V, q). We shall denote it by $\delta(V, q)$.

EXAMPLE (2.12). If $2=0$ in R then the rank of V is even and $Z(C(V,q)=R$. Furthermore we have

$$R \subset D_1(V,q)=\{y \in C_0/xy=yx, x \in V\} \subset Z(C(V,q))=R,$$

hence $D_1(V,q)=R$.

PROPOSITION (2.13). $D_1(V,q)=X(D(V,q))$.

Proof. The case where $\operatorname{rank}_R(V)$ is odd is given in [1, Remark 3.13, p. 43]. If the rank of V is even, then

$$D_1(V,q)=\{y \in C_0/xy+yx=0, x \in V\}$$

since $D_1(V,q) \subset D(V,q)=Z(C_0)$. Let σ_D be the canonical involution of $D(V,q)$. We claim that $yx=x\sigma_D(y)$ for all $x \in V$ and $y \in D(V,q)$: it suffices to check it locally. If R is local, then by (2.4) and (2.1) we have $D(V,q)=R \oplus Rz$, $z^2=z+d$, $\sigma_D(z)=1-z$ and $xz+zx=x$ for all $x \in V$. Hence $zx=x(1-z)=x\sigma_D(z)$. This implies $yx=x\sigma_D(y)$ for all $y \in D(V,q)$. Therefore, if $y \in D_1(V,q)$ we have $xy+yx=0$ and $yx=x\sigma_D(y)$, hence $x(y+\sigma_D(y))=0$, for all $x \in V$. Since V is projective of constant rank we have $T_D(y)=y+\sigma_D(y)=0$ and $y \in X(D(V,q))$. We have shown that $D_1(V,q) \subset X(D(V,q))$. To prove equality, one can assume that R is local. Then by (2.1), $X(D(V,q))$ is generated by $1-2z$. Since $xz+zx=x$, for all $x \in V$, we have $x(1-2z)+(1-2z)x=0$ and $X(D(V,q))$ is contained in $D_1(V,q)$.

Remark (2.14). For any quadratic space (V,q) of rank r, the pair $(\Lambda^r V, \det q)$ is a discriminant module. Its class $\operatorname{disc}(V,q)$ in $\operatorname{Disc}(R)$ is called in [3] the (unsigned) *discriminant* of (V,q). As shown in [14] (or by (2.4)) we have

$$\delta(V,q)=(R,(-1)^{r(r-1)/2}) \cdot \operatorname{disc}(V,q)$$

in $\operatorname{Disc}(R)$ if R is local (or a domain by (2.10)), with $\frac{1}{2} \in R$. Here (R,u), u a unity of R, denotes the discriminant module (Re,α) with $\alpha(e \otimes e)=u$. By (2.3) and (2.14) we have:

COROLLARY (2.15). *Let (V,q) be a quadratic space of rank $4n$ over a domain R such that $\frac{1}{2} \in R$. Then (V,q) has trivial* Arf *invariant if and only if (V,q) has trivial discriminant.*

3. QUATERNION ALGEBRAS

A central separable R-algebra A which is a projective R-module of rank 4 will be called here a *quaternion algebra.* By [6, p. 104] there exists a faithfully flat R-algebra S such that $S \otimes A \simeq M_2(S)$. If $\alpha: S \otimes A \simeq M_2(S)$ is such an isomorphism, we define the reduced trace T_A and the reduced norm N_A of A by $T_A(a) = \mathrm{Tr}(\alpha(1 \otimes a))$ and $N_A(a) = \det(\alpha(1 \otimes a))$, $a \in A$, where Tr is the trace and det is the determinant in $M_2(S)$. If follows by faithfully flat descent (see [6]) that T_A and N_A take values in R and are independent of the choices of α and S. Furthermore we have for any isomorphism $\beta: A \to B$ of quaternion algebras

$$T_B(\beta(x)) = T_A(x) \qquad \text{and} \qquad N_B(\beta(x)) = N_A(x), \qquad x \in A. \tag{3.1}$$

Let $\sigma_A : A \to A$ be the R-linear map defined by

$$\sigma_A(x) = T_A(x) - x, \qquad x \in A. \tag{3.2}$$

We verify immediately that, for any isomorphism $\beta: A \simeq B$,

$$\sigma_B \circ \beta = \beta \circ \sigma_A. \tag{3.3}$$

The map σ_A is an involution of A, that is,

$$\sigma_A^2 = Id_A \qquad \text{and} \qquad \sigma_A(xy) = \sigma_A(y)\, \sigma_A(x), \qquad x, y \in A. \tag{3.4}$$

The properties (3.4) are easy to check for $A = M_2(R)$. Using (3.3) they are verified for any quaternion algebra A by descent. By the same argument, we see that any element $x \in A$ satisfies its reduced characteristic polynomial

$$x^2 - T_A(x) \cdot x + N_A(x) = 0. \tag{3.5}$$

Hence we have $N_A(x) = x\sigma_A(x) = \sigma_A(x)\, x$ for all $x \in A$. The reduced norm N_A is quadratic form on A and one verifies again by descent that the pair (A, N_A) is a quadratic space, called the *norm form* of A.

EXAMPLE (3.6). Let S be a quadratic algebra with involution σ_S and let c be a unit of R. The algebra $(c, S]$ as defined in Section 2 is a quaternion algebra: recall that $A = S \oplus uS$ with the multiplication rule $xu = u\sigma_S(x)$, $x \in S$ and $u^2 = c$. The map

$$S \otimes A \to M_2(S) \qquad \text{given by } s \otimes (x + uy) \mapsto s \begin{pmatrix} x & c\bar{y} \\ y & \bar{x} \end{pmatrix},$$

where $\bar{x} = \sigma_S(x)$, is an isomorphism of S-algebras. Hence $T_A(x + uy) = x + \bar{x}$ and $N_A(x + uy) = x\bar{x} - cy\bar{y}$. Furthermore we have $\sigma_A(x + uy) = \bar{x} - uy$. If S is a free R-module with basis $\{1, z\}$ such that $z^2 = az + b$ (see (2.1)), then the matrix of the bilinear form b_A associated to N_A with respect to the basis $\{1, z, u, uz\}$ of A is

$$\left(\begin{array}{cc|cc} 2 & a & & \\ a & -2b & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & -2c & -ca \\ & & -ca & 2bc \end{array}\right).$$

PROPOSITION(3.7). *For any quaternion algebra A, the Clifford algebra of (A, N_A) is isomorphic as a graded algebra to $M_2(A)$, where $M_2(A)$ has the "chess-board" gradation.*

Proof. We define $\rho: A \to M_2(A)$ by $\rho(a) = \left(\begin{smallmatrix} 0 & \bar{a} \\ a & 0 \end{smallmatrix}\right)$, $a \in A$, where $\bar{a} = \sigma_A(a)$. Clearly $\rho(a)^2 = N_A(a)$, hence by the universal property of the Clifford algebra we have an homomorphism $C(A, N_A) \to M_2(A)$ which by (2.7) is an isomorphism.

Remark (3.8). Let λ be a unit of R. If we replace ρ by the map $a \mapsto \left(\begin{smallmatrix} 0 & \lambda\bar{a} \\ a & 0 \end{smallmatrix}\right)$, we see that also $C(A, \lambda N_A) \simeq M_2(A)$.

COROLLARY(3.9). *For any unit $\lambda \in R$, the* Arf invariant *of $(A, \lambda N_A)$ is trivial.*

Proof. Since the isomorphism (3.8) is graded, we have $C_0 \simeq A \times A$ and $Z(C_0) = R \times R$.

THEOREM (3.10). *Two quaternion algebras A and B over R are isomorphic if and only if there is a unit $\lambda \in R$ such that the quadratic spaces $\lambda \cdot N_A$ and N_B are isometric.*

Proof. By (3.1) any isomorphism $\beta: A \simeq B$ is an isometry $(A, N_A) \simeq (B, N_B)$. Conversely, let β: $(A, \lambda N_A) \simeq (B, N_B)$ be an isometry. By (3.8), β induces an isomorphism of graded algebras $M_2(A) \simeq M_2(B)$. Hence the two algebras A and B are Brauer-equivalent (see [6]). It then follows from (5.6) p. 96 of [6] that $B \simeq \mathrm{End}_A(P)$ for some projective right A-module P of rank one. Therefore we obtain a graded isomorphism γ: $\mathrm{End}_A(A \oplus A) \simeq \mathrm{End}_A(P \oplus P)$. By (3.2) of [7], γ is induced by a graded isomorphism $(A \oplus A) \otimes I \simeq P \oplus P$ for some (graded) $I \in \mathrm{Pic}(R)$. Hence $A \otimes I \simeq P$ and $A \simeq \mathrm{End}_A(A \otimes I) \simeq \mathrm{End}_A(P) \simeq B$.

4. HERMITIAN FORMS

Let A be an R-algebra with an involution $\sigma = \sigma_A$. Let $\mathcal{Mod}$-A be the category of right A-modules and A-$\mathcal{Mod}$ be the category of left A-modules. For any $M \in \mathcal{Mod}$-A, we denote M^σ the module M considered as left A-module through σ:

$$am^\sigma = (m\bar{a})^\sigma, \qquad a \in A \qquad \text{and} \qquad m \in M$$

where m^σ is m considered as element of M^σ and $\bar{a} = \sigma(a)$. Hence σ gives a functor $\mathcal{Mod}$-$A \to A$-$\mathcal{Mod}$ and symmetrically a functor A-$\mathcal{Mod} \to \mathcal{Mod}$-$A$. We have another functor $*$: $\mathcal{Mod}$-$A \to A$-$\mathcal{Mod}$ (and A-$\mathcal{Mod} \to \mathcal{Mod}$-$A$) by associating to any M the dual $M^* = \mathrm{Hom}_A(M, A)$. Recall that if M is a right A-module, then M^* is a left A-module through the rule $(a \cdot f)(m) = af(m)$, $a \in A$, $m \in M$. We show now that the two functors σ and $*$ commute. More precisely:

LEMMA 4.1. *The map* $f \mapsto Tf\colon m^\sigma \mapsto \overline{f(m)}$ *defines an isomorphism* $T_M\colon (M^*)^\sigma \to (M^\sigma)^*$ *of* A*-modules which is a natural transformation* $*\sigma \simeq \sigma *$.

Proof. To fix notations we assume that M is a right A-module. We first verify that Tf belongs to $(M^\sigma)^* = \mathrm{Hom}_A(M^\sigma, A)$:

$$Tf(am^\sigma) = Tf((m\bar{a})^\sigma) = \overline{f(m\bar{a})} = \overline{f(m)\,\bar{a}} = a\overline{f(m)} = aTf(m^\sigma).$$

Now we show that T is A-linear:

$$T(fa)(m^\sigma) = T(\bar{a}f)(m^\sigma = \overline{\bar{a}f(m)} = \overline{f(m)}\,a = T(f)\,a(m^\sigma).$$

It is clear finally that T is a natural isomorphism.

Associating with every A-module M the module $M\hat{\ } = (M^*)^\sigma$ we define a functor $\hat{\ }\colon \mathcal{M}od\text{-}A \to \mathcal{M}od\text{-}A$ (or $A\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od$). Notice that if $\varphi\colon M \to N$ is an homomorphism of A-modules, then as a map $\varphi\hat{\ } = \varphi^*$, the transposed map. Let now $\mathcal{F}\mathcal{P}\text{-}A$ be the category of the *faithfully projective* (right) A-modules. For any $P \in \mathcal{F}\mathcal{P}(A)$, the canonical map $P \to P^{**}$ given by $p \to p^{**}\colon f \to f(p)$ is an isomorphism of A-modules. By (4.1) we also have an isomorphism $P^{**} \simeq (((P^*)^\sigma)^*)^\sigma = P\hat{\ }\hat{\ }$, since $Q^{\sigma\sigma} = Q$ for any Q. Composing both, we obtain a natural transformation

$$\chi_P\colon P \to P\hat{\ }\hat{\ } \tag{4.2}$$

which is given by $p \mapsto p\hat{\ }\hat{\ }\colon f^\sigma \mapsto \overline{f(p)}$. It is easily checked that χ satisfies $\hat{\chi_P} \circ \chi_{P\hat{\ }} = \mathrm{Id}_{P\hat{\ }}$. Hence $\hat{\ }$ is a *duality functor* in the sense of [13]. We shall identify each $P \in \mathcal{F}\mathcal{P}(A)$ with $P\hat{\ }\hat{\ }$ and each homomorphism $\varphi\colon P \to Q$ with $\varphi\hat{\ }\hat{\ }$.

Let now $\varphi\colon P \simeq P\hat{\ }$ be an isomorphism of A-modules. By definition of $P\hat{\ }$, φ is a R-linear isomorphism $P \to \mathrm{Hom}_A(P, A)$ such that $\varphi(xa) = \bar{a}\varphi(x)$, $x \in P$, $a \in A$. We define a R-bilinear map $h\colon P \times P \to A$ by $h(x, y) = \varphi(x)(y)$, $x, y \in P$. We have

$$h(xa, yb) = \bar{a}h(x, y)\,b, \qquad a, b \in A, \tag{4.3}$$

hence h is a sesquilinear map. Assume now that $\varphi\hat{\ } = \varphi$. Then $h(x, y) = \varphi(x)(y) = \varphi\hat{\ }(x\hat{\ }\hat{\ })(y) = x\hat{\ }\hat{\ }(\varphi(y)) = \overline{\varphi(y)(x)} = \overline{h(y, x)}$:

$$h(x, y) = \overline{h(y, x)}, \qquad x, y \in P. \tag{4.4}$$

Conversely, if (4.4) holds for $h(x,y)=\varphi(x)(y)$, then $\varphi\hat{}=\varphi$. A R-bilinear map $h: P\times P\to A$ verifying (4.3) and (4.4) is called a *hermitian form* with values in A and the pair (P,h) or equivalently the pair (P,φ) is called a *hermitian space* if φ is an isomorphism. An isometry $\alpha: (P_1,\varphi_1)\simeq(P_2,\varphi_2)$ between two hermitian spaces (P_1,φ_1) and (P_2,φ_2) is an isomorphism $\alpha: P_1\simeq P_2$ such that $\varphi_1=\alpha\circ\varphi_2\circ\alpha$ or equivalently such that

$$h_2(\alpha(x),\alpha(y))=h_1(x,y) \qquad \text{for all } x,y\in P,$$

if h_i is the form associated with φ_i, $i=1,2$.

Let (P,φ) be a hermitian space over A and let $B=\mathrm{End}_A(P)$. The map $\sigma_B: B\to B$ defined by

$$\sigma_B(b)=\varphi^{-1}\circ b\hat{}\circ\varphi, \qquad b\in B, \tag{4.5}$$

is an involution on B. Since P is a left B-module, $P^*=\mathrm{Hom}_A(P,A)$ is a right B-module (through the action $f\cdot b=b^*f, f\in P^*$). We denote the module P^* as left B-module through σ_B also by $P\hat{}$. Then (4.5) implies that φ is also an isomorphism $P\simeq P\hat{}$ of left B-modules and we can use φ to define a hermitian form on P with values in B by

$$\tilde{h}(x,y)(z)=x(\phi(y)(z)), \qquad x,y,z\in P. \tag{4.6}$$

We recall now some results from Morita theory which we shall use later. In particular we shall see that it is equivalent to consider the hermitian structure h on P with values in A or the hermitian structure $\tilde{h}$ with values in B.

If A and B are R-algebras, we denote A-$\mathcal{M}od$-B the category of A-B-bimodules, that is of modules with a left action of A, a right action of B and two actions commute. Let $P\in B$-$\mathcal{M}od$-A and $Q\in A$-$\mathcal{M}od$-B. Assume that there exists two maps

$$f: P\otimes_A Q\to B \qquad \text{and} \qquad g: Q\otimes_B P\to A \tag{4.7}$$

such that f is a homomorphism of B-bimodules and g is a homomorphism of A-bimodules. Assume further that the maps f and g are associative in the following sense: if $f(p\otimes q)$ is denoted by pq and $g(q\otimes p)$ by qp, then $(pq)p'=p(qp')$ and $(qp)q'=q(pq')$ for all $p, p'\in P$, $q,q'\in Q$. We then have the following

THEOREM (4.8) ([2]). *If f and g are surjective maps, then f and g are isomorphisms and*

(1) *P and Q are faithfully projective A-modules and B-modules,*

(2) *the functors $P\otimes_A -$, $-\otimes_B P$, $Q\otimes_B -$ and $-\otimes_A Q$ define*

equivalences between the appropriate categories of A-(resp. B-) modules. These equivalences restrict to equivalences of the corresponding categories $\mathscr{F}\mathscr{P}$ of faithfully projective modules.

(3) *f and g induce isomorphisms as A and B-bimodules*

$$\begin{array}{ccc} \mathrm{Hom}_A(P, A) \simeq Q & & \mathrm{Hom}_B(P, B) \simeq Q \\ & \textit{and} & \\ \mathrm{Hom}_A(Q, A) \simeq P & & \mathrm{Hom}_B(Q, B) \simeq P, \end{array}$$

(4) *the homomorphisms of R-algebras*

$$\mathrm{End}_A(P) \rightleftarrows B \xrightarrow{\sim} \mathrm{End}_A(Q)^0,$$
$$\mathrm{End}_B(P)^0 \rightleftarrows A \xrightarrow{\sim} \mathrm{End}_B(Q)$$

induced by the bimodule structures of P and Q are isomorphisms.

Conversely, if $P \in \mathscr{F}\mathscr{P} - A$ *and* $B = \mathrm{End}_A(P)$, $Q = \mathrm{Hom}_A(P, A)$, *then* $P \in B\text{-}\mathscr{M}od\text{-}A$, $Q \in A\text{-}\mathscr{M}od\text{-}B$ *and the maps* $f\colon P \otimes_A Q \to B$ *and* $g\colon Q \otimes_B P \to A$ *given by* $f\colon p \otimes q \mapsto (x \mapsto pq(x))$ *and* $g(q \otimes p) = q(p)$ *are isomorphisms.*

A Morita theory for hermitian forms was developed in [5]. A special case is given by the two forms h and $\tilde{h}$ considered above: let (P, φ) be a hermitian space over A, $B = \mathrm{End}_A(P)$ and σ_B the involution (4.5). As we have seen, φ is also an isomorphism of B-modules. By Morita theory $P^\wedge$ is canonically isomorphic to the dual of P with respect to B. If $\tilde{\varphi}\colon P \xrightarrow{\sim} P^\wedge$ is a B-isomorphism which induces the given involution σ of $A = \mathrm{End}_B(P)^0$, then $\tilde{\varphi}$ is also a A-isomorphism. Hence any form $\tilde{h}$ is induced by the form h and conversely.

Another application of Morita theory is the following construction of a hermitian space from two given hermitian spaces: let $P_i \in \mathscr{F}\mathscr{P}\text{-}A$, $i = 1, 2$, and let $B_i = \mathrm{End}_A(P_i)$. Then $\mathrm{Hom}_A(P_1, P_2) \in B_2\text{-}\mathscr{F}\mathscr{P}\text{-}B_1$ and the dual of $\mathrm{Hom}_B(P_1, P_2)$ with respect to B_1 or B_2 is by Morita theory $\mathrm{Hom}_A(P_2, P_1)$. We assume now that P_1 and P_2 are hermitian spaces with isomorphisms $\varphi_i\colon P_i \xrightarrow{\sim} P_i^\wedge$, $i = 1, 2$. Let $\mathrm{Hom}_A(P_2, P_1)^\sigma$ be $\mathrm{Hom}_A(P_2, P_1)$ considered as a left B_2-, right B_1-module through the involutions σ_i of B_i induced by the φ_i's. We define a bimodule isomorphism

$$\begin{aligned} \psi\colon \mathrm{Hom}_A(P_1, P_2) &\xrightarrow{\sim} \mathrm{Hom}_A(P_1, P_2)^{*\sigma} \\ &= \mathrm{Hom}_A(P_1, P_2)^\wedge \qquad (=\mathrm{Hom}_A(P_1, P_2)^\sigma) \qquad (4.9) \end{aligned}$$

by $\psi(f) = \varphi_1^{-1} \circ f^\wedge \circ \varphi_2$, $f \in \mathrm{Hom}_A(P_1, P_2)$. We claim that ψ defines a hermitian form on $\mathrm{Hom}_A(P_1, P_2)$ with values in B_1 (or B_2 by Morita duality) or that $\psi^\wedge = \psi$: the identification

$$[\mathrm{Hom}_A(P_1, P_2)]^* = \mathrm{Hom}_A(P_2, P_1)$$

where $*$ means dualization with respect B_1 is given by

$$f \in \operatorname{Hom}_A(P_2, P_1) \mapsto (x \mapsto f(x) = f \circ x, x \in \operatorname{Hom}_A(P_1, P_2)).$$

Hence we have $\psi(f)(x) = \varphi_1^{-1} \circ f\hat{} \circ \varphi_2 \circ x$. On the other hand

$$\psi\hat{}(f\hat{}\hat{})(x) = f\hat{}\hat{}(\psi\hat{}(x)) = \overline{\psi(x)(f)}$$

by (4.2). Here $-$ means involution in B_1, therefore we have $\psi\hat{}(f\hat{}\hat{})(x) = \varphi_1^{-1} \circ (\varphi_1^{-1} \circ x\hat{} \circ \varphi_2 \circ f)\hat{} \circ \varphi_1 = \varphi_1^{-1} \circ f\hat{} \circ \varphi_2 \circ x = \psi(f)(x)$ using that $\hat{\varphi_i} = \varphi_i$. Let

$$h_1(x, y) = \psi(x) \circ y \qquad \text{and}$$

$$h_2(x, y) = x \circ \psi(y), \qquad x, y \in \operatorname{Hom}_A(P_1, P_2) \tag{4.10}$$

be the two corresponding hermitian forms with values in B_1, resp. B_2. Using the notation (4.6) we have $\tilde{h}_1 = h_2$ and $\tilde{h}_2 = h_1$. Let (P_3, φ_3) be a third hermitian space and let $f \in \operatorname{Hom}_A(P_1, P_2)$ and $g \in \operatorname{Hom}_A(P_2, P_3)$. We have with some obvious abuses of notations

$$\overline{g \circ f} = \bar{f} \circ \bar{g} \tag{4.11}$$

if $\bar{f} = \psi(f)$ and ψ is defined as in (4.9) (for different modules!).

We shall later consider projective A-modules of rank one, where A is a central separable R-algebra. In this case the rank has the following interpretation: Let $P \in \mathscr{F}\mathscr{P} - A$. Since A is a finitely generated projective R-module, P is also a projective R-module and we say that P has rank r over A if $\operatorname{rank}_R P = r \cdot \operatorname{rank}_R A$. We shall use later the following "splitting principle":

LEMMA (4.12). *Let A be a central separable R-algebra and let P be a right A-module. Then the following properties are equivalent*:

(1) *P is a projective A-module of rank r, where r is a positive integer.*

(2) *There exists a faithfully flat R-module S such that $S \otimes P \simeq (S \otimes A)^r$ as right $S \otimes A$-modules.*

Proof. (2) $\Rightarrow$ (1) is evident by faithfully flat descent. So let P be a projective of rank r, r a positive integer, over A. The algebra A can be split by a faithfully flat R-algebra S, i.e., there is an isomorphism $\beta: S \otimes A \simeq \operatorname{End}_S(V)$ of S-algebras where $V \in \mathscr{F}\mathscr{P} - S$ (see [6]). Consider now $S \otimes P$ as right $\operatorname{End}_S(V)$-module by β^{-1}. By (4.8), $S \otimes P$ is isomorphic to a module $M \otimes_S V^*$ as right $\operatorname{End}_S(V)$-module, where $\operatorname{End}_S(V)$ acts on $V^* = \operatorname{Hom}_S(V, S)$ and M is a faithfully projective S-module. Counting ranks shows that $\operatorname{ranks}_S M = r \cdot \operatorname{rank}_S V$. Hence there is a faithfully flat S-algebra

T such that $T \otimes_S M = T \otimes_S V^r$. Replacing S by T we can as well assume that $M \simeq V^r$ as S-modules. Therefore

$$S \otimes P \simeq V^r \otimes_S V^* \simeq \mathrm{End}_S(V)^r$$

as right $\mathrm{End}_S(V)$-modules and we have $S \otimes P \simeq (S \otimes A)^r$ as claimed.

5. Hermitian Forms and Quadratic Spaces

Let A be a quaternion algebra in the sense of Section 3, this means that A is a central separable of rank 4 and let σ_A be the canonical involution of A (see Section 3). Let $\mathscr{F}\mathscr{P}_1 - A$ be the category of faithfully projective right A-modules of rank one. If $P \in \mathscr{F}\mathscr{P}_1 - A$ then $B = \mathrm{End}_A(P)$ is again a quaternion algebra. The canonical involution σ_B of B can be interpreted as an isomorphism $B \simeq B^0$ of B with the opposite algebra B^0. By Morita theory (see (4.8)) $B^0 \simeq \mathrm{End}_A(P^*) \simeq \mathrm{End}_A(P^\wedge)$. Composing both isomorphisms we have an isomorphism

$$\psi_P : \mathrm{End}_A(P) \simeq \mathrm{End}_A(P^\wedge) \tag{5.1}$$

which by (3.1) of [7] is induced by an isomorphism $\varphi_P : P \otimes I_P \simeq P^\wedge$, for some $I_P \in \mathrm{Pic}(R)$, that is:

$$\psi_P(x) = \varphi_P \circ x \otimes 1 \circ \varphi_P^{-1}, \qquad x \in \mathrm{End}_A(P). \tag{5.2}$$

By definition of ψ_P we have

$$\sigma_B(x) = \varphi_P^{-1} \circ x^\wedge \circ \varphi_P, \qquad x \in \mathrm{End}_A(P) \tag{5.3}$$

using that $\mathrm{End}_R(I_P) \simeq R$. We shall call $P \in \mathscr{F}\mathscr{P}_1 - A$ *special* if $I_P \simeq R$. Then φ_P is an isomorphism $P \simeq P^\wedge$. Clearly, φ_P is only defined up to a unit of R. Conversely if φ and φ' are isomorphisms $P \simeq P^\wedge$ inducing σ_B as in (5.3), then $\varphi^{-1} \circ \varphi'$ lies in the center of B, hence is a unit of R. We claim that any special A-module (P, φ_P) is a hermitian space, which means that $\varphi_P^\wedge = \varphi_P$. Since, as we have seen, two maps φ and φ' which induce σ_B as in (5.3) differ by a unit of R, it suffices to verify $\varphi^\wedge = \varphi$ for any φ inducing σ_B. By (4.12) and descent, we can assume that P is a free A-module of rank one. The claim is then nearly immediate:

Example (5.4). If $P = eA$ is a free A-module with basis element e, then A can be identified with $B = \mathrm{End}_A(P)$ through left multiplication. Let $e^\wedge = e^*$ be the dual basis (e^* as element of P^* and $e^\wedge$ as element of $P^\wedge$). Then φ given by $\varphi(ea) = e^\wedge a$ induces $\sigma_A = \sigma_B$ and clearly $\varphi^\wedge = \varphi$.

We sum up these results in the following proposition:

PROPOSITION (5.5). *Let $P \in \mathscr{F}\mathscr{P}_1 - A$. If P is special, i.e., the canonical involution of $B = \mathrm{End}_A(P)$ is induced by an isomorphism $\varphi_P : P \simeq P\hat{}$, then (P, φ_P) is an hermitian space. The isomorphism φ_P then is unique up to a unit of R.*

We shall call a hermitian space as in (5.5) *special.*

COROLLARY(5.6). *If* $\mathrm{Pic}(R) = 0$, *then any $P \in \mathscr{F}\mathscr{P}_1 - A$ carries the structure of a special hermitian space (P, φ_P).*

For any special hermitian space (P, φ_P) we define a quadratic form N_P by

$$N_P(x) = \varphi_P(x)\, x, \qquad x \in P. \tag{5.7}$$

The form N_P has values in R: by descent we assume that P is free, then by (5.4) we can choose φ_P such that $\varphi_P(ea) = e\hat{}\, a$. Hence $N_P(ea) = \varphi(ea)(ea) = \bar{a}\varphi(e)(e)\, a = \bar{a}a = N_A(a) \in R$. The induced bilinear form b_P is the composition of the hermitian form $h_P : P \times P \to A$ with the reduced trace T_A, $b_P = T_A \circ h_p$. Using that the bilinear form $(x, y) \mapsto T_A(xy)$ on A is non-singular, it is easy to check that b_P also is non-singular. Hence the pair (P, N_P) is a quadratic space. Notice that N_P is only defined up to a unit of R by the special module structure of P. We shall call (P, N_P) the *reduced norm* of P. As we have seen, the reduced norm of a free module eA is just the reduced norm of A. It follows from Section 4 of [7] that N_P coincide with the reduced norm defined in [7]. Remark that if we consider (P, φ_P) as an hermitian space over $B = \mathrm{End}_A(P)$ and if $\tilde{N}_P$ is the corresponding quadratic space, then $\tilde{N}_P = N_P$. This is easy if P is free and the general case is a consequence of the splitting principle (4.12).

Let (P_1, φ_1) and (P_2, φ_2) be two special hermitian spaces over A. We denote $\mathrm{Hom}_A(P_1, P_2)$ by $[P_1, P_2]$. As we have seen in (4.9) the map $\varphi_{P_1P_2} : [P_1, P_2] \to [P_1, P_2]\hat{}$ defined by $\varphi_{P_1P_2}(f) = \varphi_1^{-1} \circ f\hat{} \circ \varphi_2$ is a hermitian structure on $[P_1, P_2]$ over $B_1 = \mathrm{End}_A(P_1)$ (or $B_2 = \mathrm{End}_A(P_2)$). By Morita theory $B_2 = \mathrm{End}_{B_1}([P_1, P_2])$ and one can check that $[P_1, P_2]$ is actually a special hermitian space. Let $N_{P_1P_2}$ be the corresponding quadratic form. Notice that

$$N_{P_1P_2} = \tilde{N}_{P_1P_2} = N_{P_2P_1}.$$

EXAMPLE (5.8). We have $N_{PP} = N_B$, where $B = \mathrm{End}_A(P)$ and $N_{AP} = N_P = N_{P\hat{}}$ if we choose $\varphi_A = \mathrm{Id}$ as in Example (5.4).

If (P_i, φ_i), $i = 1, 2, 3$, are three special hermitian spaces, then, it follows from (4.11) that we have the composition law

$$N_{P_1P_3}(g \circ f) = N_{P_1P_2}(f) \cdot N_{P_2P_3}(g) \tag{5.9}$$

for $f \in [P_1, P_2]$ and $g \in [P_2, P_3]$. An immediate consequence of (5.9) and (4.12) is:

PROPOSITION (5.10). *Let (P_1, φ_1) and (P_2, φ_2) be two special hermitian spaces over A. Then for any homomorphism $\beta: P_1 \to P_2$ of A-modules we have $h_{P_2}(\beta(x), \beta(y)) = N_{P_1P_2}(\beta) \cdot h_{P_1}(x, y)$, where h_{P_i} is the hermitian form associated with φ_i, and $N_{P_2}(\beta(x)) = N_{P_1P_2}(\beta) \cdot N_{P_1}(x)$. If β is an isomorphism, then $N_{P_1P_2}(\beta)$ is a unit of R.*

Proposition (5.10) implies that if the two modules P_1 and P_2 are isomorphic, then there is a unit $\lambda \in R$ such that the hermitian forms $(P_1, \lambda h_{P_1})$ and (P_2, h_{P_2}), resp. the quadratic space $(P_1, \lambda N_{P_1})$ and (P_2, N_{P_2}), are isometric. We shall now prove a converse. For this we need to compute the Clifford algebra of the norm $N_{P_1P_2}$:

PROPOSITION (5.11). *For any unit $\lambda \in R$, the Clifford algebra C of the quadratic space $([P_1, P_2], \lambda N_{P_1P_2})$ is isomorphic as a graded algebra to* $\mathrm{End}_A(P_1 \oplus P_2)$ *where* $\mathrm{End}_A(P_1 \oplus P_2)$ *has the "chess-board" gradation.*

Proof. We write the elements of $E = \mathrm{End}_A(P_1 \oplus P_2)$ as matrices

$$\begin{pmatrix} b_1 & y \\ x & b_2 \end{pmatrix}$$

where $b_i \in B_i = \mathrm{End}_A(P_i)$, $x \in [P_1, P_2]$ and $y \in [P_2, P_1]$. Then the map $\rho: [P_1, P_2] \to E$ given by

$$\rho(x) = \begin{pmatrix} 0 & \lambda\varphi_{P_1P_2}(x) \\ x & 0 \end{pmatrix}, \qquad x \in [P_1, P_2],$$

satisfies $\rho(x)^2 = \lambda N_{P_1P_2}(x)$ in E. Hence ρ induces a graded homomorphism $C \to E$ which by (2.7) is an isomorphism.

Remark (5.12). Proposition (3.8) is in fact a special case of (5.11).

COROLLARY (5.13). *The quadratic space $([P_1, P_2], \lambda N_{P_1P_2})$ has trivial Arf invariant for any unit λ of R.*

Proof. We have $C_0 \simeq B_1 \times B_2$ and $Z(C_0) \simeq R \times R$.

THEOREM (5.14). *Let (P_i, φ_i) be special hermitian spaces with reduced norms N_{P_i}, $i = 1, 2$. If there is a unit $\lambda \in R$ such that the quadratic spaces $(P_1, \lambda N_{P_1})$ and (P_2, N_{P_2}) are isometric, then there exists a projective R-module I of rank one such that $4 \cdot [I] = 0$ in* $\mathrm{Pic}(R)$ *and such that $P_1 \otimes I$ and P_2 are isomorphic A-modules.*

Proof. Let $\beta: P_1 \simeq P_2$ be an isometry. By (5.8) and (5.11) there exists a graded isomorphism $\beta': \mathrm{End}_A(A \oplus P_1) \simeq \mathrm{End}_A(A \oplus P_2)$ which extends β (where P_i is identified with $\mathrm{Hom}_A(A, P_i)$ in $\mathrm{End}_A(A \oplus P_i)$). By (3.1) of [7], β' is induced by an isomorphism $f: (A \oplus P_1) \otimes I \simeq A \oplus P_2$ for some $I \in \mathrm{Pic}(R)$, i.e., $\beta'(x) = fxf^{-1}$. Let $f = f_0 + f_1$ be the decomposition of f in graded elements. Since $\beta(x)f = f(x)$ for all $x \in P_1$ and since x and $\beta(x)$ have degree one we have $\beta(x)f_0 = f_0 x$ and $\beta(x)f_1 = f_1 x$. It then follows that $f^{-1}f_0$ is in the center of $\mathrm{End}_A(A \oplus P_1)$, hence is an element $\rho \in R$. If ρ is not zero, then f has degree zero. If ρ is zero, then f has degree one. Hence f is in any case graded. Hence we have

$$\begin{matrix} A \otimes I \simeq A \\ P_1 \otimes I \simeq P_2 \end{matrix} \qquad \text{or} \qquad \begin{matrix} A \otimes I \simeq P_2 \\ P_1 \otimes I \simeq A. \end{matrix}$$

In the second case we have $P_1 \otimes I \otimes I \simeq P_2$ and since $P_2 \simeq \hat{P_2}$ also $A \otimes I \otimes I \simeq A$. If J is a projective R-module of rank one such that $A \otimes J \simeq A$ as right A-modules, then it follows from (2.4) of [7] that $2[J] = 0$ in $\mathrm{Pic}(R)$. Hence we have $4[I] = 0$ in $\mathrm{Pic}(R)$ for both cases.

COROLLARY (5.15). *Let R be a ring such that* $\mathrm{Pic}(R)$ *has no two torsion. Then P_1 and P_2 are isomorphic if and only if there is a unit $\lambda \in R$ such that the quadratic spaces $(P_1, \lambda N_{P_1})$ and (P_2, N_{P_2}) are isometric.*

Proof. One direction is a consequence of (5.10) and the other of (5.14).

Remark (5.16). Let α be an automorphism of A and let ${}_\alpha A_1$ be the bimodule A where the action on the left is through α. As shown in [6, IV, Section 1], if $A \otimes J \simeq A$ as right A-module, for some J of rank one over R, then there exists an automorphism α of A such that $A \otimes J \simeq {}_\alpha A_1$, as bimodules. Hence the proof of (5.14) shows that there is an automorphism α of A and an α-semilinear isomorphism $P_1 \simeq P_2$ if N_{P_1} and N_{P_2} are isometric.

6. QUADRATIC SPACES WITH TRIVIAL ARF INVARIANT

PROPOSITION (6.1). *Let (V, q) be a quadratic space of even rank and let $C = C_0 \oplus C_1$ be the Clifford algebra of (V, q). Then the following properties are equivalent*:

(1) *The Arf invariant of (V, q) is trivial.*

(2) *There exists an idempotent $e \in C$ such that $xe + ex = x$ for all $x \in V$.*

(3) *There exists an idempotent $e \in C$ such that*

$$C_0 = \{x \in C / xe = ex\} \qquad \textit{and} \qquad C_1 = \{x \in C / xe + ex = x\}.$$

Proof. (1) $\Rightarrow$ (2). If $a(V,q)=0$, then $D=D(V,q)=Z(C_0)\simeq R\times R$ which implies the existence of an idempotent $e\in D$ such that $D=R\oplus Re$. We can now assume that R is local to show $xe+ex=x$ for all $x\in V$ since it suffices to verify it locally. By (2.4), $D=R\oplus Rz$ with $z^2=z+d$ and z satisfies $zx+xz=x$ for all $x\in V$. We write $e=\alpha z+\beta$, $\alpha,\beta\in R$ and α is a unit of R. From $e^2=e$ and $z^2=z+d$ we deduce $\alpha=1-2\beta$, hence $e=(1-2\beta)z+\beta$. This implies that $xe+ex=(1-2\beta)(zx+xz)+2\beta x=x$, for all $x\in V$.

(2) $\Rightarrow$ (1). Let $e\in C$ be an idempotent such that $xe+ex=x$ for all $x\in V$. Then $xye=exy$ for all $x,y\in V$, hence e lies in $Z(C_0)=D$. We show that $D=R\oplus Re$. We first notice that 1 and e are linearly independent over R: if $\lambda_1+\lambda_2e=0$, then $\lambda_2xe=\lambda_2ex$, hence $2\lambda_2xe=\lambda_2x$ or $\lambda_2x(1-2e)=0$. Since $(1-2e)^2=1$ we have $\lambda_2x=0$, for all $x\in V$. Since V is finitely generated of positive rank, this implies $\lambda_2=0$ and $\lambda_1=0$. We now have $R\oplus Re\subset D$ and since $e^2=e$, $R\oplus Re$ is a subalgebra of D. To check that $R\oplus Re=D$ we can again assume that R is local. By (2.4), $D=R\oplus Rz$ with $z^2=z+d$ and $zx+xz=x$, for all $x\in V$. Hence $(z-e)x+x(z-e)=0$ and $(1-2e)x+x(1-2e)=0$. Since C is central and V generates C, we see that the element $\beta=(z-e)(1-2e)$ lies in R. Then $z-e=(z-e)(1-2e)^2=\beta(1-2e)$ and $z=(1-2\beta)e+\beta$ lies in $R\oplus Re$. This shows that $D=R\oplus Re$.

(2) $\Rightarrow$ (3). It is easy to check that (2) implies $C_0\subset\{x\in C/xe=ex\}$ and $C_1\subset\{x\in C/xe+ex=x\}$ (see the beginning of the proof of (2) $\Rightarrow$ (1)). We show that $C_0\supset\{x\in C/xe=ex\}$, the corresponding inclusion for C_1 can be proved in a similar way. Let $x=x_0+x_1\in C_0+C_1$ be such that $xe=ex$. Then we have $x_1e=ex_1$ and $x_1=x_1e+ex_1=2ex_1$ or $(1-2e)x_1=0$. Since $(1-2e)^2=1$, we must have $x_1=0$. Finally (3) $\Rightarrow$ (2) is trivial since $V\subset C_1$.

Remark (6.2). An idempotent $e\in C$ satisfying (3) is called in [11] an idempotent which defines the gradation. Hence (6.1) means that the Clifford algebra of a quadratic space of even rank has an idempotent which defines the gradation if and only if (V,q) has trivial Arf invariant.

We shall now apply (6.1) to describe the Clifford algebra of a quadratic module of rank $4r$ with trivial Arf invariant. We first need a slight generalization of the notion of an isometry of hermitian spaces: let A and A' be algebras with involutions which are isomorphic, $\alpha\colon A\simeq A'$, and let (P,φ), resp. (P',φ'), be a hermitian space over A, resp. A'. We shall say that (P,φ) and (P',φ') are *semi-isometric* if there is a α-semilinear isomorphism $\beta\colon P\simeq P'$ such that $\varphi=\hat{\beta}\circ\varphi'\circ\beta$.

THEOREM (6.3). *Let (V,q) be a quadratic space of rank $4r$ with trivial Arf invariant and let C be the Clifford Algebra of (V,q). Then there exist a* *central separable R-algebra A of rank 2^{4r-2} and a projective right A-module P of rank one such that:*

(1) *$C \simeq \mathrm{End}_A(A \oplus P)$ as graded algebras, where the gradation of $\mathrm{End}_A(A \oplus P)$ is the chess-board gradation.*

(2) *The canonical involution σ_C of C induces by restriction*

(i) *involutions σ_A and σ_B on A and $B = \mathrm{End}_A(P)$;*

(ii) *an isomorphism $\varphi: P \simeq P^\wedge$ such that $\sigma_B(b) = \varphi^{-1} \circ b^\wedge \circ \varphi$, $b \in B$, and $\varphi^\wedge = \varphi$. Hence (P, φ) is a hermitian space over A (or B) and the involution of B (or of A) is induced by the hermitian structure of P.*

(3) *If A' is a central separable R-algebra and (P', φ') is a hermitian space over A' verifying (1) and (2), then either $A' \simeq A$ or $A' \simeq \mathrm{End}_A(P)$ and $A \simeq \mathrm{End}_{A'}(P')$ and in the both cases the hermitian spaces (P, φ) and (P', φ') are semi-isometric.*

Proof. By (6.1) we have an idempotent e in $D(V, q) \subset C$ which defines the gradation. Let $f = 1 - e$, $A = eCe$ and $P = fCe$. Then A is an Re-algebra (Re has e as unit element), hence an R-algebra through the isomorphism $R \simeq Re$. The map $eCe \to \mathrm{End}_C(eC)$ induced by the left eCe structure of eC is an isomorphism with inverse $\alpha \to \alpha(e) = \alpha(e)\, e$, $\alpha \in \mathrm{End}_C(eC)$. Hence A is central separable, since C is central separable. The module P is obviously a right A-module. Similarly let $B = fCf$ and $Q = eCf$; B is central separable and Q is a right B-module. The obvious maps $P \otimes_A Q \to B$ and $Q \otimes_B P \to A$ are surjective and they satisfy the associativity relations of the maps (4.7). It follows from (4.8) that $P \in B - \mathscr{F}\mathscr{P} - A$ and $Q \in A - \mathscr{F}\mathscr{P} - B$ and that $Q \simeq P^* = \mathrm{Hom}_A(P, A)$ and $B \simeq \mathrm{End}_A(P)$. To show that P has rank one over A it suffices to verify that A and B have the same rank over R. This is however clear since it is true over a field by [4, Section 9, No. 4]. Since

$$\mathrm{End}_{eCe}(Ce) = \mathrm{End}_{eCe}(eCe + fCe) = \mathrm{End}_A(A \oplus P),$$

the map $C \to \mathrm{End}_{eCe}(Ce)$ induced by the left C-structure of Ce gives an homomorphism of algebras $C \to \mathrm{End}_A(A \oplus P)$ which is an isomorphism by (2.7). To prove that it is an isomorphism of graded algebras, we first note that by (6.1), $A = eCe = eC_0$, $B = fCf = fC_0$, $P = fCe = fC_1 = C_1 e$ and $P^* = Q = eCf = eC_1 = C_1 f$. This implies that $(\mathrm{End}_A(A \oplus P))_0 = A \oplus B = eC_0 + fC_0 = C_0$ and that $(\mathrm{End}_A(A \oplus P))_1 = P \oplus P^* = C_1 e + C_1 f = C_1$. We now verify (2): by (2.5) the canonical involution σ_C of C induces the identity on $D(V, q)$, hence $\sigma_C(e) = e$ and $\sigma_C(f) = f$. Therefore σ_C induces by restriction an involution σ_A on A, an involution σ_B on B and a bijective map $\varphi: P = fCe \to eCf = P^*$. One verifies easily that φ is such that $\varphi(xa) = \bar{a}\varphi(x)$ where $\bar{a} = \sigma_A(a)$, hence φ is an isomorphism of right A-modules $P \simeq P^\wedge$ (see Section 4). We check that $\sigma_B(b) = \varphi^{-1} \circ b^\wedge \circ \varphi$. To simplify notations we write $\sigma_C(c)=\bar{c}$, $c \in C$. We have $\varphi^{-1} \circ b^{\hat{}} \circ \varphi(x) = \varphi^{-1}(b^{\hat{}}\bar{x}) = \varphi^{-1}(\bar{x}b)$ since $b^{\hat{}} = b^*$. Furthermore $\varphi^{-1}(\bar{x}b) = \bar{b}x$, hence $\bar{b} = \varphi^{-1} \circ b^{\hat{}} \circ \varphi$. Finally we verify that $\varphi^{\hat{}} = \varphi$: y definition $\varphi^{\hat{}}(p) = \varphi^{\hat{}}(p^{\hat{}\hat{}})$ and $\varphi^{\hat{}}(p^{\hat{}\hat{}}) = p^{\hat{}\hat{}}\varphi$ since $\varphi^{\hat{}} = \varphi^*$. Hence

$$\begin{aligned}\varphi^{\hat{}}(p)(x) &= p^{\hat{}\hat{}}\varphi(x) = \overline{\varphi(x)(p)} = \overline{\varphi(x)p} = \bar{p}\overline{\varphi(x)} = \bar{p}\bar{\bar{x}}\\ &= \bar{p}x = \varphi(p)(x)\end{aligned}$$

since $\varphi(p) = \sigma_C(p) = \bar{p}$.

The uniqueness statement (3) follows from Morita theory as in the proof of (5.4.3) in [7].

We now apply (6.3) to the case where (V, q) has rank 4. Then the algebra A has rank 4, hence is a quaternion algebra as defined in Section 3. We claim that the involution σ_A of (6.3.2) is the canonical involution of A. It suffices to verify it locally and so we can assume that V is the orthogonal sum of two planes,

$$V = (Re_1 \oplus Re_3) \perp (Re_2 \oplus Re_4)$$

as in Example (2.4). Let L be the subalgebra of A generated by $i = ee_1e_2$ and $j = ee_1e_4$. Since $i^2 = -\alpha_1\alpha_2 e$, $j^2 = -\alpha_1\beta_2 e$ and $ij + ji = -\alpha_1 e$, L is the Clifford algebra over Re of the quadratic form $q(\lambda_1 i + \lambda_2 j) = -\lambda_1^2\alpha_1\alpha_2 e - \lambda_1\lambda_2\alpha_1 e - \lambda_1^2\alpha_1\beta_2 e$, which is nonsingular since the associated bilinear form has the matrix

$$\begin{pmatrix} -2\alpha_1\alpha_2 e & -\alpha_1 e \\ -\alpha_1 e & -2\alpha_1\beta_2 e \end{pmatrix},$$

whose determinant is $-\alpha_1^2(1 - 4\alpha_2\beta_2)\, e$, and $1 - 4\alpha_2\beta_2$ and α_1 are units of R (see (2.4)). It follows from (1.2) that L is central separable, hence by (2.7) $L = A$. Let $x \mapsto \bar{x}$ be the canonical involution of L (or A!). Since $\bar{i} = -i$ and $\bar{j} = -j$ we see that the canonical involution and σ_A coincide. Similarly σ_B is the canonical involution of the quaternion algebra B.

Now we claim that $P = Ve$, hence $P \simeq V$ as R-module since the map $V \to Ve$, $v \mapsto ve$, is an isomorphism of R-modules. We have $C_1 = V + Vz$: this can be shown locally using the basis $\{e_1, e_2, e_3, e_4\}$ as above and z as in (2.4). Hence $C_1 = V + Ve$ since $\{1, z\}$ and $\{1, e\}$ both generate $D(V, q)$. This implies that $P = fCe = C_1 e = Ve$ as claimed. By (6.3.2), (P, φ) is a special hermitian space and the quadratic form $\varphi(x)(x)$, $x \in P$ is the reduced norm N_P of P. Using $P = Ve$, we see that $N_P(ve) = \varphi(ve)(ve) = evve = q(v)\, e$ and (P, N_P) and (V, q) are isometric. To sum up we have the existence part of the following classification theorem:

THEOREM (6.4). *Let (V, q) be a quadratic space of rank 4 with trivial Arf invariant. Then there exist a quaternion algebra A, a special hermitian space (P, φ) over A such that $(V, q) \simeq (P, N_P)$ where N_P is the reduced norm on P. If (P', φ') is another special hermitian space with values in a quaternion algebra A' such that $(V, q) \simeq (P', N_{P'})$, then either $A' \simeq A$ or $A' \simeq \mathrm{End}_A(P)$ and $A \simeq \mathrm{End}_{A'}(P')$ and in both cases (P, φ) and (P', φ') are semi-isometric.*

The uniqueness part of (6.4) follows from (6.3.3) since by (5.11) we have $C(P', N'_P) \simeq \mathrm{End}_{A'}(A' \oplus P')$.

Remark (6.5). It is clear that the two cases in the uniqueness statement of (6.4) must appear, since any hermitian space (P, φ) over A is automatically a hermitian space over $\mathrm{End}_A(P)$.

Remark (6.6). Theorem (5.14) is a consequence of the uniqueness part of (6.4) if we take Remark (5.16) in account.

Remark (6.7). In the Theorem (6.3) we have associated with any quadratic space (V, q) of rank $4r$ and trivial Arf invariant a hermitian space (P, φ) over a central separable R-algebra A with involution. Part (1) of (6.3) suggests that the algebra $E = \mathrm{End}_A(A \oplus P)$ may be of some interest for any hermitian space (P, φ). We first notice that E has an involution given by the hermitian structure of $A \oplus P$. If we write any element of E as a matrix $\left(\begin{smallmatrix} a & q \\ p & b \end{smallmatrix}\right)$ with $a \in A$, $b \in B = \mathrm{End}_A(P)$, $p \in \mathrm{Hom}_A(A, P) = P$ and $q \in P^{\hat{}}$ $(=P^*$ as a set), then the involution is given by

$$\begin{pmatrix} a & q \\ p & b \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{p} \\ \bar{q} & \bar{b} \end{pmatrix},$$

where $a \mapsto \bar{a}$ is the involution on A, $b \to \bar{b} = \varphi^{-1} \circ b^{\hat{}} \circ \varphi$ is the involution on B, $\bar{p} = \varphi(p)$ and $\bar{q} = \varphi^{-1}(q)$. The injective maps $i: P \to E$ and $j: P \to E$ defined by

$$i(p) = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \quad \text{and} \quad j(p) = \begin{pmatrix} 0 & \bar{p} \\ 0 & 0 \end{pmatrix}, \quad p \in P,$$

have the property that $\overline{i(x)}\, i(y) = h(x, y) = \varphi(x)(y)$ and $j(x)\overline{j(y)} = \tilde{h}(x, y) = x\varphi(y)$. The pair (E, i) can be characterized by the following universal property: let (P, h) be a hermitian space over A. Then for any algebra with involution C, any homomorphism $\gamma: A \to C$ of algebras with involutions and any γ-semilinear map $\gamma': P \to C$ such that $\overline{\gamma'(x)}\, \gamma'(y) = \gamma(h(x, y))$, $x, y \in P$, there exist a unique homomorphism $\gamma'': E \to C$ of algebras with involutions such that γ'' extends γ and $\gamma'' \circ i = \gamma'$. We define $\gamma''|_A = \gamma$, $\gamma''|_P = \gamma'$ and $\gamma''|_{P^*} = \sigma \circ \gamma' \circ \varphi^{-1}$ (where σ is the involution of E). To define $\gamma''|_B$ we use that $B = P \otimes_A P^*$ and put $\gamma''(p \otimes f) = \gamma'(p)\, \sigma \circ \gamma' \circ \varphi^{-1}(f)$. Notice that we also have $\gamma''(x)\overline{\gamma''(y)} = \gamma''(\tilde{h}(x, y))$, $x, y \in P$. In fact we have $E \simeq \mathrm{End}_B(B \oplus P)$ and there is a complete symmetry between A and B, h and $\tilde{h}$, i and j. We call E the *enveloping algebra* of h (or $\tilde{h}$). As we have seen, if (P_i, φ_i), $i = 1, 2$ are hermitian spaces over A, then $\mathrm{Hom}_A(P_1, P_2)$ is an hermitian space over B_1 or B_2 where $B_i = \mathrm{End}_A(P_i)$ and it is easy to verify that the enveloping algebra of $\mathrm{Hom}_A(P_1, P_2)$ is $\mathrm{End}_A(P_1 \oplus P_2)$.

7. Hermitian Spaces over Quaternion Algebras and Quadratic Algebras

Let A be a quaternion algebra of the form $(c, S]$ (see (3.6)) and let (P, φ_P) be a special hermitian space over A (see (5.5)). We shall see that there is a hermitian structure on P with values in S "between" the hermitian structure over A and the associate quadratic structure over R. We recall that A is of the form $A = S \oplus uS$, where $u^2 = c$ is a unit of R, S is a quadratic R-algebra and $xu = u\sigma(x)$, $x \in S$. We have a trace map, which is S-linear:

$$T_{A/S}: A \to S \tag{7.1}$$

given by $T_{A/S}(x + uy) = x$. Note that $T_A = T_S \circ T_{A/S}$. Any right A-module M is also a right S-module through scalar restriction. Let $M' = \mathrm{Hom}_S(M, S)$ considered as right S-module through the involution σ_S of S (which is the restriction of the involution of A, as we have seen in (3.6)). The map $T: M^\wedge \to M'$ given by $f \mapsto T_{A/S} \circ f$ is S-linear. We have:

Lemma (7.2). *The homomorphism $T: P^\wedge \to P'$ is an isomorphism of S-modules for any $P \in \mathscr{F}\mathscr{P} - A$.*

Proof. By the "splitting principle" (4.12) we can assume that P is a free A-module. Since T commute with finite direct sums, we can even assume that P is a free A-module of rank one with basis element e. Let $e^\wedge \in P^\wedge$ be the dual basis element. Then

$$T(e^\wedge a)(eb) = T_{A/S}(\bar{a}b),$$

hence we have to check that the map $\alpha_A: A \to A'$ given by $a \mapsto (b \mapsto T_{A/S}(\bar{a}b))$ is an isomorphism. If we take $e_1 = 1$ and $e_2 = u$ as basis of A over S, with $\{e_1', e_2'\}$ the dual basis, then α_A is given by the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -c\end{smallmatrix}\right)$, which is invertible. Therefore α_A is an isomorphism.

Let (P, φ_P) be a special hermitian space over A with form $h_P: P \times P \to A$. Then $q_P = \mathrm{Tr}_{A/S} \circ h_P: P \times P \to S$ is a hermitian form over S. Since q_P is induced by the homomorphism

$$\alpha_P = T_{A/S} \circ \varphi_P: P \to P'$$

which by (7.2) is an isomorphism, the form q_p is nonsingular. Furthermore we have $q_P(x, x) = N_P(x)$, $x \in P$. We sum up in the following proposition:

PROPOSITION (7.3). *Let (P, φ_P) be a special hermitian space over $A = (c, S]$. Then (P, α_P), with $\alpha_P = T_{A/S} \circ \varphi_P$, is a hermitian space of rank two over S. Furthermore $\alpha_P(x)(x) = N_P(x)$, $x \in P$, where N_P is the reduced norm of P.*

Let (P_1, φ_1) and (P_2, φ_2) be two special hermitian spaces over $A = (c, S]$. Then we have as in (5.10):

$$q_{P_2}(\beta(x), \beta(y)) = N_{P_1P_2}(\beta) \cdot q_{P_1}(x, y), \qquad x, y \in P_1, \tag{7.4}$$

for any homomorphism $\beta\colon P_1 \to P_2$ of A-modules and we can conclude as in (5.15) *that*:

THEOREM (7.5). *Let R be a ring such that* Pic(R) *has no 2-torsion and let A be a quaternion R-algebra of the form $(c, S]$, S a quadratic R-algebra. Then any projective right A-module P of rank one carries a non-singular hermitian form q_P with values in S. Two such modules P_1 and P_2 are isomorphic if and only if there is a unit $\lambda \in R$ such that the two hermitian forms λq_{P_1} and q_{P_2} are isometric.*

Remark (7.6). For R an $\mathbb{R}$-algebra and $A = \mathbb{H} \otimes_{\mathbb{R}} R$ ($\mathbb{H}$ the real quaternions) Parimala used in [12] cohomological methods to associate a hermitian space of rank two to any projective A-module of rank one. Her construction was generalized by Kopeiko in [8] to arbitrary quaternion algebras $H = ((a, b)/k)$ over a field k and polynomial rings R. It can be checked that the hermitian form q_P gives the forms of Parimala and Kopeiko in these cases.

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